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LETTER TO THE EDITOR

A second class of solvable potentials related to the Jacobi polynomials

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Abstract Recently, Lévai has used a simple method for generating solvable wave equations to construct potentials whose solutions include as factors Jacobi polynomials of functions $g(x)$ which solve specific differential equations Application of this method to a previously unconsidered differential equation involving Jacobi polynomials results in a new class of solvable potentials missing from earlier compilations An interesting property of these new potentials is that when two of their defining parameters are fixed, they all possess the same energy eigenvalue spectrum independent of any change in a third parameter, leading to an indefinite number of different potentials with the same energies

With the advent of supersymmetric quantum mechanics and shape invariance, interest in exact solutions of single dimensional wave equations has increased markedly. Lévai (1989, 1991), building on the work of Dabrowska (1988), has suggested a simple method for generating exactly solvable single dimensional wave equations involving orthogonal polynomials. Generally, any function $F(g(x))$ satisfying the second-order differential equation

$$\frac{d^2F(g)}{dg^2}+Q(g)\frac{dF(g)}{dg}+R(g)F(g)=0 \tag{1}$$

can serve as the basis for a wavefunction

$$\Psi(x)=f(x)F(g(x)) \tag{2}$$

satisfying

$$\frac{d^2\Psi}{dx^2}+(E-V(x))\Psi=0 \tag{3}$$

where $f(x)$ is given by

$$f(x)=(g')^{-1/2}\exp\left[\int^g Q(x)\,dg\right] \tag{4}$$

and the second term in the wave equation is given by

$$E-V(x)=R(g(x))(g')^2-(f''/f). \tag{5}$$

In terms of g, (5) can be written

$$E-V(x)=\frac{g'''}{2g'}-\frac{3}{4}\left[\frac{g''}{g'}\right]^2+(g')^2\left[R(g)-\frac{1}{2}\frac{dQ}{dg}-\frac{1}{4}Q^2(g)\right]. \tag{6}$$

0305-4470/91/120667+04\$03 50

Lévai's approach has been to determine the $Q(g)$ and $R(g)$ corresponding to specific orthogonal polynomials, and to set at least one of the terms on the right in (6) to a constant to represent E This leads to differential equations in $g(x)$ of the form

$$(g')^2 H(g) = \text{constant}, C \tag{7}$$

where $H(g)$ depends on the particular term in (6) under consideration. Once suitable $g(x)$ have been found in this manner, the parameters in (6) are redefined such that the potential $V(x)$ is no longer dependent on the index n of the orthogonal polynomial, which serves as the quantum number for the re-parametrized potential.

For the Jacobi polynomials $P_n^{\alpha,\beta}(g(x))$, Lévai considered the differential equations

$$(g')^2/(1-g^2) = C \qquad (g')^2/(1-g^2)^2 = C \qquad (g')^2 g/(1-g^2)^2 = C \tag{8}$$

obtaining the $g(x)$ corresponding to his type of solutions PI, PII and PIII. However, for Jacobi polynomials, the function

$$F(g) = (1-g)^{(\alpha+1)/2}(1+g)^{(\beta+1)/2} P_n^{\alpha,\beta}(g) \tag{9}$$

satisfies (1) where

$$Q(g) = 0$$

$$R(g) = \frac{1}{4}\frac{(1-\alpha^2)}{(1-g)^2} + \frac{1}{4}\frac{(1-\beta^2)}{(1+g)^2} + \frac{2n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)}{2(1-g^2)} \tag{10}$$

(Abramowitz and Stegun, 1970, equation 22.6.3). For $Q(g)$ and $R(g)$ in (10), application of (6) results in

$$E - V(x) = \frac{g'''}{2g'} - \frac{3}{4}\left[\frac{g''}{g'}\right]^2 + \frac{1}{4}(1-\alpha^2)\frac{(g')}{(1-g)^2} + \frac{1}{4}(1-\beta^2)\frac{(g')^2}{(1+g)^2}$$
$$+ \frac{[2n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)]}{2}\frac{(g')^2}{(1-g^2)} \tag{11}$$

and, following the procedure outlined, the previously unconsidered equations

$$\frac{(g')^2}{(1-g)^2} = C \tag{12}$$

and

$$\frac{(g')^2}{(1+g)^2} = C \tag{13}$$

arising from the first two terms in (11).

Consider a solution $g(x)$ of (12). Defining a function $h(x) = -g(x)$ shows that $h(x)$ solves (13). Hence for the purpose of finding solvable potentials, only one of these two differential equations need be considered. One general solution of (13) is $g(x) = a\exp(-x) - 1$, where a is an arbitrary constant. Substitution into (11) gives

$$E - V(x) = +\frac{1}{4}(1-\alpha^2)\frac{a^2\exp(-2x)}{(2-a\exp(-x))^2} - \frac{\beta^2}{4}$$
$$+ \frac{[2n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)]}{2}\frac{a\exp(-x)}{(2-a\exp(-x))} \tag{14}$$

and the potential $V(x)$ is dependent on the quantum number n while $E = -\beta^2/4$.

The dependence of $V(x)$ on the quantum number n can be removed as follows. Set

$$\beta=\frac{A+(\alpha+1)(\alpha-1)/4}{n+(\alpha+1)/2}-[n+(\alpha+1)/2]. \tag{15}$$

Then

$$2n\beta=2A+(\alpha-1)(\alpha+1)/2-\frac{(\alpha+1)A}{n+(\alpha+1)/2}-\frac{(\alpha+1)^2(\alpha-1)/4}{n+(\alpha+1)/2}-2n^2-n(\alpha+1) \tag{16}$$

and

$$(\alpha+1)(\beta+2)=\frac{(\alpha+1)A}{n+(\alpha+1)/2}+\frac{(\alpha+1)^2(\alpha-1)/4}{n+(\alpha+1)/2}-(\alpha+1)^2/2+(\alpha+1). \tag{17}$$

Combining (16) and (17) gives

$$2n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)=2A \tag{18}$$

and substitution into (14) gives

$$V(x)=\tfrac{1}{4}(1-\alpha^2)\frac{a^2\exp(-2x)}{(2-a\exp(-x))^2}+\frac{Aa\exp(-x)}{2-a\exp(-x)} \tag{19}$$

and a potential independent of n. From (2), (4) and (10),

$$\Psi(x)\approx[(a\exp(-x))^{\beta/2}(2-a\exp(-x))^{(\alpha+1)/2}]P_n^{\alpha,\beta}(a\exp(-x)-1) \tag{20}$$

where $\Psi(x)$ is not normalized, and α and β are defined as above.

In order to have $\Psi(x)$ which are square integrable, some conditions need to be imposed on the possible values of α and β in (20). From the definition of the Jacobi polynomials (Abramowitz and Stegun 1970), the initial restriction is that $\alpha, \beta > -1$. This condition on α ensures that the second factor in $\Psi(x)$ remains finite for positive x. The extra condition imposed on β by (20) is that β remain positive. From (15) this leads to the extra condition that the quantum number n satisfies

$$n^2+(n+\tfrac{1}{2})(\alpha+1)<A. \tag{21}$$

This equation has several implications for the interrelationship of α, A and n. Basically, for a given α, there is a minimum value of A required before any $n>0$ can be obtained, likewise for fixed α and A, there are limits set on the maximum value of n. In terms of α, A and n, the energy eigenstates are given explicitly as

$$E=-\frac{1}{4}\left(\frac{A+(\alpha^2-1)/4-[n+(\alpha+1)/2]^2}{n+(\alpha+1)/2}\right)^2. \tag{22}$$

The potentials $V(x)$ and wavefunctions $\Psi(x)$ can be viewed as generalizations of the Hulthén potential (Hulthén 1942), where $\alpha=1$ and $a=2$. Usually, however, the polynomial in $\Psi(x)$ for this case is not recognized as a Jacobi polynomial (e.g. Flügge 1971). An extension of the Hulthén potential, sometimes referred to as the Manning–Rosen–Newing potential (Manning and Rosen 1933, Newing 1935, 1940), retains $a=2$ but no longer requires $\alpha=1$. Solutions to this latter potential are discussed by Myhrman (1980) and Boudjedaa *et al* (1991). Myhrman explicitly notes the relationship of $\Psi(x)$

and the Jacobi polynomials in his equation (34). With α, A and a as variable parameters subject to the conditions above, the potential $V(x)$ can be either a totally repulsive potential or an anharmonic potential well. The Manning–Rosen–Newing potential has attracted attention mainly as a way to generalize the Hulthén potential to states of higher angular momentum l.

This new class of solvable potentials would appear to be of interest on two grounds. First, they may serve as models for single variable anharmonic potentials, such as the potentials describing diatomic electronic states. Varshni (1957) previously considered the Manning–Rosen–Newing potentials in this case, and concluded that the Morse potential gave the best all-around results. However, different values for the parameters could possibly give improved results. Second, of greater theoretical interest is the result that for fixed α and A, the energy eigenvalues are identical. Supersymmetric quantum mechanics (e.g. Sukumar, 1985) can be used to generate partner potentials for a Hamiltonian H which remove the ground state, add a state below the original ground state, or maintain the original eigenvalues. Given that supersymmetric quantum mechanics predicts a result demonstrated in this new class, its supersymmetric properties bear further investigation.

References

Abramowitz M and Stegun I A 1970 *Handbook of Mathematical Functions* (New York: Dover)
Boudjedaa T *et al* 1991 *J. Math. Phys.* **32** 441
Dabrowska J, Khare A and Sukhatme U 1988 *J. Phys. A: Math. Gen.* **21** L195
Flugge S 1971 *Practical Quantum Mechanics* (Berlin: Springer)
Hulthén L 1942 *Ark. Mat. Astron. Fys.* A **28** No 5
Lévai G 1989 *J. Phys. A: Math. Gen.* **22** 689
—— 1991 *J. Phys. A: Math. Gen.* **24** 131
Manning M F and Rosen N 1933 *Phys. Rev.* **44** 953
Myhrman U 1980 *J. Math. Phys.* **21** 1732
Newing R A 1935 *Phil. Mag.* **19** 759
—— 1940 *Phil. Mag.* **29** 298
Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2917
Varshni V P 1957 *Rev. Mod. Phys.* **29** 664